Assignment :- 1

- Multiple choice questions

1] Real part of the complex number $z = e^{5+i\frac{\pi}{2}}$ is

c) 0

2] polar form of complex number $z = x+iy$ is

a) $r(\cos\theta + i\sin\theta)$

3] If $z = -1+i$ then arg (z) is equal to

c) $\pi - \frac{\pi}{4}$

4] $\int \cosh x \, dx$ is equal to

a) $\sinh x$

5] The differential equation of the form $\frac{dx}{dy} + px = q$ where p and q are function of y or constants, is

c) linear differential eq^n in x

6] The linear differential equation $(1+y^2) + (x - e^{-\tan^{-1}y})\frac{dy}{dx} = 0$ has intigrating factor

c) $e^{\tan^{-1}y}$

Date:

7] Using Demoivre's theorem, exponential form of the expression $\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{1000}$

8] The general value of logarithm of complex number $\log(1+i\sqrt{3})$ is

9] The differential equation of the form $\frac{dy}{dx} + py = q$ where p and q are function of x or constants, is

b) linear differential equation

10] The differential equation of the form $\frac{dy}{dx} + py = qy^n$ $n \neq 1$ where p and q are function of x or constants is,

- Solve

1] Simplify using Demoivre's theorem, the expression
$$\frac{(\cos 2\theta - i\sin 2\theta)^7 (\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 4\theta + i\sin 4\theta)^{12} (\cos 5\theta - i\sin 5\theta)^{-6}}$$

→ Here,

$\cos 2\theta - i\sin 2\theta = \cos(-2\theta) + i\sin(-2\theta)$
$= (\cos\theta + i\sin\theta)^{-2}$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^{3}$

$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^{4}$

$\cos 5\theta - i\sin 5\theta = \cos(-5\theta) + i\sin(-5\theta)$
$= (\cos\theta + i\sin\theta)^{-5}$

from the eq^n

$$= \frac{[(\cos\theta + i\sin\theta)^{-2}]^{7}[(\cos\theta + i\sin\theta)^{3}]^{-5}}{[(\cos\theta + i\sin\theta)^{4}]^{12}[(\cos\theta + i\sin\theta)^{-5}]^{-6}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-14}(\cos\theta + i\sin\theta)^{-15}}{(\cos\theta + i\sin\theta)^{48}(\cos\theta + i\sin\theta)^{30}}$$

$$= (\cos\theta + i\sin\theta)^{-14-15-48-30}$$

$$= (\cos\theta + i\sin\theta)^{-107}$$

$$= \cos(-107\theta) + i\sin(-107\theta)$$

$$= \cos(107\theta) + i\sin(107\theta)$$

2) Find modulus and principal value of the argument of

$$\left(\frac{(1+i\sqrt{3})^{13}}{(\sqrt{3}-1)^{11}}\right)$$

→ $z = \frac{(1+i\sqrt{3})^{13}}{(\sqrt{3}-i)^{11}}$

$1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\sqrt{3}-i = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$= \frac{\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{13}}{\left[2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]^{11}}$$

$$= \frac{2^{13}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{13}}{2^{11}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{11}}$$

$$= \frac{(2)^2\left(\cos^{13}\frac{\pi}{3} + i\sin^{11}\frac{\pi}{3}\right)}{\left(\cos\frac{11\pi}{3} - i\sin\frac{11\pi}{3}\right)}$$

$$= 4\left[\left(\cos\frac{13\pi}{3} + i\sin\frac{13\pi}{3}\right)\left(\cos\frac{11\pi}{6} - i\sin\frac{11\pi}{6}\right)^{-1}\right]$$

$$= 4\left[\left(\cos\frac{13\pi}{3} + i\sin\frac{13\pi}{3}\right)\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)\right]$$

$$= 4\left[\cos\left(\frac{13\pi}{3} + \frac{11\pi}{6}\right) + i\sin\left(\frac{13\pi}{3} + \frac{11\pi}{6}\right)\right]$$

$= 4\left[\cos\frac{37\pi}{6} + i\sin\frac{37\pi}{6}\right]$

$= 4\left[\cos\left(6\pi + \frac{\pi}{6}\right) + i\sin\left(6\pi + \frac{\pi}{6}\right)\right]$

$= 4\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

by compairing with $z = r(\cos\theta + \sin\theta)$

$r = 4$ & $\theta = \frac{\pi}{6}$

3] Solve the eq^n $x^3 - 1 = 0$

→

$x^3 - 1 = 0 \quad \therefore x^3 = 1$

$\therefore x^3 = \cos 0 + i\sin 0$

$x = (\cos 0 + i\sin 0)^{1/3}$

$= \{\cos(2n\pi + 0) + i\sin(2n\pi + 0)\}^{1/3}$

$= \cos\frac{(2n\pi + 0)}{3} + i\sin\frac{(2n\pi + 0)}{3}$

$= \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$

For the root of given eq^n put $n = 0, 1, 2$

$1 = x_0 = \cos 0 + i\sin 0 = 1$

$\omega = x_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$= \omega$

$\omega^2 = x_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$= -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

$= \omega^2$

4] For $x = \sqrt{3}$ Find values of $\tanh(\log x)$

→ We know,

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\therefore \tanh(\log x) = \frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}$

$= \frac{e^{\log x} - e^{\log x^{-1}}}{e^{\log x} + e^{\log x^{-1}}}$

$= \frac{x - x^{-1}}{x + x^{-1}}$

$= \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{\frac{x^2 - 1}{x}}{\frac{x^2+1}{x}} = \frac{x^2 - 1}{x^2 + 1}$

put $x = \sqrt{3}$

$$= \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^2 + 1} = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$

5] P.T $\cosh^5 x = \frac{1}{16} (\cos 5x + 5 \cosh 3x + 10 \cosh x)$

→

$\cos 5x + 5 \cosh 3x + 10 \cosh x$

$$\cosh^5 x = \left(\frac{e^x - e^{-x}}{2}\right)^5$$

$$= \frac{1}{(2)^6} (e^x - e^{-x})^5$$

$$= \frac{1}{32} (e^x + e^{-x})^5$$

$$= \frac{1}{32} (e^{5x} + 5e^{4x} e^{-x} + 10e^{3x} e^{-2x} + 10e^{2x} e^{-3x} + 5e^x \cdot e^{-4x} + e^{-5x})$$

$$= \frac{1}{36} \left[\left(\frac{e^{5x} + e^{-5x}}{2}\right) + 5\left(\frac{e^{3x} + 3e^{-3x}}{2}\right) + 10\left(\frac{e^x + e^{-x}}{2}\right)\right]$$

$$= \frac{1}{36} [\cosh 3x + 5 \cosh 3x + 10 \cosh x]$$

6] P.T $\tan\left\{i \log \frac{a-ib}{a+ib}\right\} = \frac{2ab}{a^2 - b^2}$

→

$$\tan\left\{i \log \frac{a-ib}{a+ib}\right\}$$

Date:

$= \tan[i\{\log(a-ib) - \log(a+ib)\}]$

$= \tan[i\{\log\sqrt{a^2+b^2} - i\tan^{-1}\frac{b}{a}\}]$

$[i\{\log\sqrt{a^2+b^2} + i\tan^{-1}\frac{b}{a}\}]$

$= \tan[i\{\frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\frac{b}{a}\}]$

$-i\{\frac{1}{2}\log(a^2+b^2 + i\tan^{-1}\frac{b}{a}\}$

$= \tan[\frac{1}{2}i\log(a^2+b^2) + \tan^{-1}\frac{b}{a} - \frac{1}{2}i\log(a^2+b^2) - \tan^{-1}\frac{b}{a}$

$= \tan(2\tan^{-1}\frac{b}{a})$

$= \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}} = \frac{2ab}{a^2-b^2}$

--- Hence proved.

7] $\frac{dy}{dx} = \frac{2x-3y+1}{3x+4y-5}$

$(3x+4y-5)dy = (2x-3y+1)dx$

$\therefore (2x-3y+1)dx + (-3x+4y-5)dy = 0$

$M = 2x - 3y + 1$ $\quad N = -3x + 4y - 5$

$\frac{\partial M}{\partial y} = -3$ $\quad \frac{\partial N}{\partial x} = -3$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ The given DE is exact

$\therefore$ G.S is $\int (2x - 3y + 1)\,dx + \int (-4y - 5)\,dy = c$

$\frac{2x^2}{2} - 3xy + x - \frac{4y^2}{2} - 5y = c$

$x^2 - 3xy + x - 2y^2 + 5y = c$

8] $(1 + y^2) + (x - e^{-\tan^{-1} y}) \frac{dy}{dx} = 0$

Multiply by $\frac{dx}{dy}$ on both sides

$(1 + y^2) \frac{dx}{dy} + (x - e^{-\tan^{-1} y}) = 0$

Divided by $1 + y^2$ on both sides

$\frac{dx}{dy} + \frac{1}{1+y^2} (x) = e^{-\tan^{-1} y}$

We know,

$$\frac{dx}{dy} + px = Q$$

$$\therefore p = \frac{1}{1+y^2} \quad ; \quad Q = \frac{e^{-\tan y}}{1+y^2}$$

$$I.F = e^{\int p\,dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ G.S is $\quad x(I \cdot F) = \int Q\,(I \cdot F)\,dy + c$

$$x e^{\tan^{-1} y} = \int \frac{e^{-\tan^{-1} y}}{1+y^2} e^{\tan^{-1} y}\,dy + c$$

$$x e^{\tan^{-1} y} = \int \frac{1}{1+y^2}\,dy + c$$

$$\therefore x e^{\tan^{-1} y} = \tan^{-1} y + c$$

9] Simplify using Demoivre's theorem, the expression

$$\frac{(\cos 3\theta + i\sin 3\theta)^8 (\cos 4\theta - i\sin 4\theta)^{-2}}{(\cos 2\theta - i\sin 2\theta)^4 (\cos\theta + i\sin\theta)^3}$$

→

We know that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

$$= \frac{[\cos(24\theta) + i\sin(24\theta)][\cos(-8\theta) - i\sin(-8\theta)]}{[\cos(8\theta) - i\sin(8\theta)][\cos(3\theta) + i\sin(3\theta)]}$$

$$= \frac{[\cos(24\theta) + i\sin(24\theta)][\cos(8\theta) + i\sin(8\theta)]}{[\cos(8\theta) - i\sin(8\theta)][\cos(3\theta) + i\sin(3\theta)]}$$

We know that $\cos\theta + i\sin\theta = e^{i\theta}$

$\cos\theta - i\sin\theta = e^{-i\theta}$

$$= \frac{e^{24\theta} \cdot e^{8\theta}}{e^{-8\theta} \cdot e^{3\theta}}$$

$$= e^{24+8+8-3}$$

$$= e^{37}$$

$\therefore [\cos\theta + i\sin\theta]^{37}$

i.e. $[\cos(37\theta) + i\sin(37\theta)]$

10] Solve $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

→ Given :- $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

$$\frac{dy}{dx} = -\left[\frac{1+y^2}{1+x^2}\right]$$

Date:

$$\frac{dy}{1+y^2} = \frac{-dx}{1+x^2}$$

Integrating both sides

$$\int \frac{dy}{1+y^2} = -\int \frac{dx}{1+x^2}$$

We know that

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\tan^{-1}y = -\tan^{-1}x + c$$

$$\therefore \tan^{-1}x + \tan^{-1}y = c$$

11] Solve $\frac{dy}{dx} + \frac{2x-3y+1}{3x+4y-5}$

→

$$\frac{dy}{dx} = \frac{2x-3y+1}{3x+4y-5}$$

$$(3x+4y-5)\,dy = (2x-3y+1)\,dx$$

$$3x\,dy + 4y\,dy - 5dy = 2x\cdot dx - 3y\cdot dx + 1dx$$

$$3x\,dy + 4y\,dy - 5dy - 2x\,dx + 3y\,dx - 1dx = 0$$

$$3(x\,dy + y\,dx) + 4y\,dy - 5dy - 2x\,dx - 1dx = 0$$

$$3d(xy) + 4y\,dy - 5dy - 2x\,dx - 1dx = 0$$

$$3\int d(xy) + 4\int y\,dy - 5\int dy - 2\int x\,dx - 1\int dx = 0$$

$$3xy + \frac{4y^2}{2} - 5y - \frac{2x^2}{2} - 1x = c$$

$$3xy + 2y^2 - 5y - x^2 - 1x = c$$

$$\therefore \underline{3xy + 2y^2 - 5y - x^2 - 1x = c}.$$

12] Solve $y\frac{dy}{dx} = \sqrt{1+x^2+y^2+x^2y^2}$

→

Given differential eqⁿ is

$$\frac{y}{x}\frac{dy}{dx} = \sqrt{1+x^2+\underbrace{y^2+x^2+y^2}_{y^2(1+x^2)}}$$

$y^2(1+x^2)$ — Taking y^2 common

$$\frac{y}{x}\frac{dy}{dx} = \sqrt{\underbrace{(1+x^2)+y^2(1+x^2)}_{(1+y^2)(1+x^2)}}$$

$$\frac{y}{x}\frac{dy}{dx} = \sqrt{\underbrace{(1+y^2)(1+x^2)}_{\sqrt{1+y^2}\sqrt{1+x^2}}}$$

$\because \sqrt{ab} = \sqrt{a}\cdot\sqrt{b}$

$$\frac{y}{x}\frac{dy}{dx} = \sqrt{1+y^2}\,\sqrt{1+x^2}$$

$y\,dy = x\sqrt{1+x^2}\,dy$ — seperating variable

This is variable seperable form

$$\int \frac{y}{\sqrt{1+y^2}}\,dy = \int x\left(\sqrt{1+x^2}\right)dx \quad \text{—①}$$

Now, consider LHS of eqⁿ ①

$$\int \frac{y}{\sqrt{1+y^2}}\,dy = \frac{1}{2}\int \frac{2y}{\sqrt{1+y^2}}\,dy$$

- divide & multiply by 2

Using Integration formula

$$\left[\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}\right]$$

$$\int \frac{y}{\sqrt{1+y^2}}\,dy = \frac{1}{2}\,2\sqrt{1+y^2}$$

$$\int \frac{y}{\sqrt{1+y^2}}\,dy = \sqrt{1+y^2} \quad \text{— ②}$$

Also RHS, $= \int x\left(\sqrt{1+x^2}\right)dx$

$$\left[\begin{array}{l} \because \text{Put } 1+x^2 = t \\ 2x\,dx = dt, \quad x\,dx = \frac{dt}{2} \end{array}\right]$$

$$\int x\left(\sqrt{1+x^2}\right)dx = \int \sqrt{t}\,\frac{dt}{2}$$

$$= \frac{1}{2} \int \sqrt{t}\, dt$$

$$\int t^{1/2}\, dt \qquad \because \sqrt{t} = t^{1/2}$$

Using Scalar Integration

$$\left[\int k f(x)\, dx = k \int f(x)\, dx\right]$$

$$\int x\left(\sqrt{1+x^2}\right) dx = \frac{1}{2} \int t^{1/2}\, dt$$

$$\frac{t^{1/2+1}}{1/2+1} = \frac{t^{3/2}}{\frac{3}{2}}$$

Using Standard Integration formula

$$\because \left[\int x^n\, dx = \frac{x^{n+1}}{n+1}\right]$$

$$\int x\sqrt{1+x^2}\, dx = \frac{1}{2} \cdot \frac{t^{3/2}}{3/2} = \frac{1}{3} t^{3/2}$$

$$\int x\sqrt{1+x^2}\, dx = \frac{1}{3}(1+x^2)^{3/2} \text{ --- } (3) \quad [\because t = (1+x^2)]$$

from eq^n (2) & (3)

eq^n (1) becomes

$$\sqrt{1+y^2} = \frac{1}{3}(1+x^2)^{3/2} + c$$

13) Solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

→ Given

$$xy - \frac{dy}{dx} = y^3 e^{-x^2}$$

Multiply the given eq^n by

$$\frac{dy}{dx} - xy = -y^3 e^{-x^2}$$

This is Bernoulli's LDE

Divide this eq^n by y^3

$$\frac{1}{y^3}\frac{dy}{dx} - xy^{-2} = -e^{-x^2} \quad \text{—— (1)}$$

put $y^{1-3} = t$

$y^{-2} = t$

Diff w.r.t. x

$$-2y^{-3}\frac{dy}{dx} = \frac{dt}{dx}$$

we can also write it has

$$\frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dt}{dx}$$

eq^n ① becomes

$$-\frac{1}{2}\frac{dt}{dx} - xt = -e^{-x^2}$$

Multiply by -2 on both sides

$$\frac{dt}{dx} + 2xt = 2e^{-x^2}$$

This is LDE compare

$$\frac{dy}{dx} + Py = Q$$

$$P = 2x \quad , \quad Q = 2e^{-x^2}$$

$$I = e^{\int P dx}$$

$$= e^{\int 2x dx}$$

$$= e^{\frac{2 x^2}{2}}$$

$$I.F = e^{x^2}$$

Solution of LDE is $(t)\, I.F = \int Q \, IF \, dx$

$$\frac{1}{y^2} e^{x^2} = \int 2e^{-x^2} \cdot e^{x^2} dx$$

$$\frac{1}{y^2} e^{x^2} = \int 2 dx$$

$$\therefore \frac{e^{x^2}}{y^2} = 2x + c$$